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Finite Element Model with Nonviscous Damping

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This paper develops a procedure by which many different structural damping functions (linear or nonlinear) may be incorporated in a structure through finite element matrices and also develops a solution technique for the resulting nonlinear equations of motion. Damping is incorporated through general strain and strain-rate dependent terms in the material constitutive law; general finite element matrices are derived through application of the principle of virtual work; and the solution technique is developed by modifying the Newmark method to accommodate an iterative solution and to treat nonlinear damping as a pseudoforce. In single-degree-of-freedom problems for three types of nonviscous damping, the solution technique is described as accurate in comparison with closed-form and numerical solutions while converging on the average in one to two iterations depending on the damping model. In a four-degree-of-freedom finite element cantilevered beam problem, the solution technique is seen to be as accurate as a Gear method numerical technique while being obtained in 90% less CPU time. Finally, the application of the analytical developments of the paper are demonstrated by investigating the effects of a specific nonviscous damping model on the transient motion of a free-free Timoshenko beam. The analysis approach appears to be an effective analytical tool in the investigation of damping effects on structures, though experimental verification is still needed.

Introduction

THIS paper is concerned with the dynamic response of structures having damping dominated by internal structural damping mechanisms. Although structural damping is often negligible compared to damping due to air friction and friction in joints, structural damping can be of major importance in structures having heavy damping treatments or in outer-space structures. When structural damping is the dominant damping mechanism, its nonlinearity must be considered since there is experimental evidence that shows structural damping to be basically a nonlinear nonviscous phenomenon.[1-5] Although linear viscous or hysteretic damping models give adequate results in many engineering applications, nonlinear nonviscous models are needed to provide realistic representations of structural damping in free-vibration or nonperiodic-vibration problems.[6,7]

To solve a structural dynamics problem, a common procedure is to discretize the structure using finite element methods and then solve the resulting equations involving mass, stiffness, and damping matrices by numerical techniques. Damping matrices in the past typically have been based on viscous or hysteretic damping with the damping matrix usually being proportional to the stiffness matrix or to a linear combination of the mass and stiffness matrices. No damping matrices having sufficient generality to model the variety of nonlinear models of interest could be found in the literature. Therefore, one objective of the present study was to develop a procedure by which many different structural damping functions, viscous or nonviscous, can be incorporated in a structure through finite element damping matrices.

Numerical integration techniques for solving the set of ordinary differential equations arising from the finite element method have had much investigation in the past. However, while solution techniques for linear problems have been developed for over 25 years, solution techniques for nonlinear problems have been seriously studied only in recent years, even though many vibration problems are nonlinear.[8-13] General purpose nonlinear analysis computer programs have been developed recently to solve equations such as those derived herein, but these general purpose programs have been found to be cost prohibitive and in need of further development.[11,14,15] Thus, there is a clear need for more efficient nonlinear solution algorithms. One important approach for reducing costs is that of matching solution techniques to problem classes.[16,17] Most nonlinear finite element analyses have been concerned with statics problems, and most of the nonlinear problems studied (static or transient) have dealt with nonlinearities due to large displacement, large strain, or nonlinear stress-strain laws. Little past work has been found involving the solution of finite element problems with nonviscous damping. Therefore, the second objective of this study was to develop and evaluate an efficient solution technique for problems with nonviscous damping. The accuracy and convergence characteristics of the method developed are studied to illustrate the method's computation benefits for nonviscous damping models.

The paper first considers incorporating nonviscous damping in a structure through the use of the finite element method. Damping is included in the model through the stress-strain constitutive law by the addition of strain-rate dependent terms. Then, through the use of the principle of virtual work, the general equations for representing damping matrices are derived. To illustrate the approach the general equations are specialized for a finite element model of a Timoshenko beam.

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Next, the paper describes the solution technique used to solve the set of nonlinear equations arising from the finite element method. The solution technique is referred to as the "pseudoforce Newmark method" because it is an application of the Newmark method[18] with the nonlinear damping cast as a pseudoforce. Because the problem here is nonlinear, the classical Newmark method, a solution technique for linear problems, is modified to accommodate an iteration loop. Then the paper investigates the accuracy and convergence characteristics of the pseudoforce Newmark method in single- and multiple-degree-of-freedom problems with various forms of nonviscous damping. Finally, the solution technique is applied to a free-free Timoshenko beam finite element problem representing the motion of a long flexible member of a space structure to illustrate the effects of a specific nonviscous damping model.

Finite Element Formulation of Nonviscous Damping Matrix

Structural Damping

Structural damping (also known as internal damping, material damping, and internal friction) is defined as the energy dissipation mechanism within the volume of a vibrating solid that is independent of any dissipation capacity at the boundaries of the solid. For a continuous system this damping can be represented in a constitutive law as a stress that is a function of strain and strain rate.[19,20] The constitutive law takes the form

$$\{\sigma\} = [E]\{\epsilon\} + [\bar{E}]\{\dot{\epsilon}\} + [\bar{e}(\{\epsilon\},\{\dot{\epsilon}\})]\{\dot{\epsilon}\} \tag{1}$$

where $[E]$ is a matrix containing Young's and shear moduli, $[\bar{E}]$ a matrix containing viscous damping constants, and $[\bar{e}(\{\epsilon\}, \{\dot{\epsilon}\}]$ a matrix containing nonviscous damping functions of $\{\epsilon\}$ and $\{\dot{\epsilon}\}$. The basis for representing damping as a relationship between stress and strain rate is Newton's law of viscosity used in fluid mechanics.[21] This constitutive law is used in the finite element formulation to allow the modeling of a variety of damping functions in a structure. Many nonviscous single-degree-of-freedom damping models have been proposed previously.[2,3] The three damping models used in single-degree-of-freedom problems discussed later in this paper are described graphically in Fig. 1 in comparison to viscous damping. Type I damping,[22] a piecewise linear function, has a damping force whose magnitude is proportional to displacement but whose sign is negative or positive depending on the sign of velocity. Type II damping[2] is a quadratic function and is often called "quadratic damping." Type II damping force has a magnitude proportional to the square of velocity and a sign that goes by velocity. Type III damping[2] is a cubic function with a damping force whose magnitude is quadratically proportional to displacement while simultaneously being linearly proportional to velocity and having the sign of velocity. All the models shown in Fig. 1 have a damping force that opposes velocity. These force-displacement-velocity equations are converted to stress-strain-strain rate relations to obtain damping models for a material constitutive law. In particular, type II damping was employed in the beam problems discussed later in this paper.

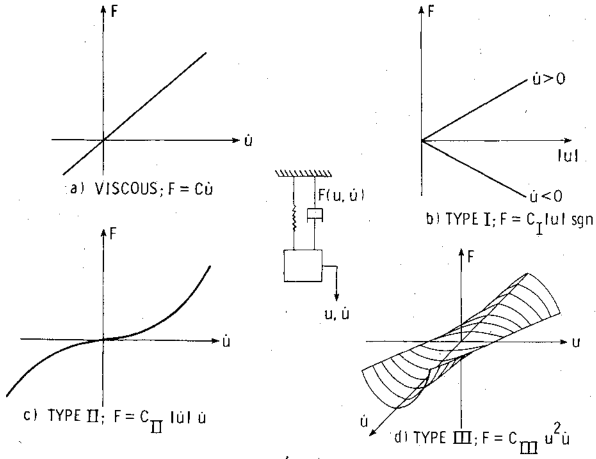

Fig. 1 Viscous and nonviscous damping models.

General Finite Element Procedure

The finite element formulation is based on the equation relating generalized displacements $\{f\}$ at any point in the element to the generalized nodal displacements $\{u\}$,

$$\{f\} = [N]\{u\} \tag{2}$$

where $[N]$ is called the displacement field matrix; the equation relating strain $\{\epsilon\}$ to $\{u\}$,

$$\{\epsilon\} = [B]\{u\} \tag{3}$$

where $[B]$ is called the strain-displacement matrix; the constitutive law given in Eq. (1); and Newton's second law of motion,

$$\{R\} = \rho\{\ddot{f}\} \tag{4}$$

where $\{R\}$ contains the inertial forces per unit volume and ρ is the mass density.

To obtain the finite element equations of motion, Eqs. (1-4) are substituted into the principle of virtual work[23] which can be written as

$$\iiint_V \{\delta\sigma\}^T\{\delta\epsilon\}\mathrm{d}V + \iiint_V \{R\}^T\{\delta f\}\mathrm{d}V = \{F(t)\}^T\{\delta u\} \tag{5}$$

where V is the volume of the element, $\{\delta\epsilon\}$ the virtual strain, $\{\delta f\}$ the virtual displacement, $\{F(t)\}$ the externally applied nodal forces, and $\{\delta u\}$ the virtual displacement through which $\{F(t)\}$ does virtual work. Substituting Eqs. (1-4) into Eq. (5) results in

$$\left[\iiint_V [B]^T[E][B]\mathrm{d}V\right]\{u\} + \left[\iiint_V [B]^T[\bar{E}][B]\mathrm{d}V\right]\{\dot{u}\}$$
$$+ \left[\iiint_V [B]^T[\bar{e}(\{\epsilon\},\{\dot{\epsilon}\})][B]\mathrm{d}V\right]\{\dot{u}\}$$
$$+ \left[\iiint_V \rho[N]^T[N]\mathrm{d}V\right]\{\ddot{u}\} = \{F(t)\} \tag{6a}$$

which is the equation of motion for the element. Equation (6a) is then rewritten as

$$[K]\{u(t)\} + [C]\{\dot{u}(t)\} + [C_{NL}(\{\dot{u}(t)\},\{u(t)\})]\{\dot{u}(t)\}$$
$$+ [M]\{\ddot{u}(t)\} = \{F(t)\} \tag{6b}$$

where

$$[K] = \text{stiffness matrix} = \iiint_V [B]^T[E][B]\mathrm{d}V \tag{7}$$

$$[C] = \text{viscous damping matrix} = \iiint_V [B]^T[\bar{E}][B]\mathrm{d}V \tag{8}$$

$$[C_{NL}(\{u\},\{\dot{u}\})] = \text{nonviscous damping matrix}$$
$$= \iiint_V [B]^T[\bar{e}(\{\epsilon\},\{\dot{\epsilon}\})][B]\mathrm{d}V \tag{9}$$

$$[M] = \text{mass matrix} = \iiint_V \rho [N]^T [N] \mathrm{d}V \tag{10}$$

The element matrices are all symmetric because $[E]$, $[\bar{E}]$, and $[\bar{e}(\{\epsilon\}, \{\dot{\epsilon}\})]$ are symmetric. Note that $[C_{NL}]$ is written as a function of $\{u\}$ and $\{\dot{u}\}$ because the matrix $[\bar{e}(\{\epsilon\}, \{\dot{\epsilon}\})]$ occurs in the integral definition of $[C_{NL}]$ and $\{\epsilon\} = [B]\{u\}$.

Timoshenko Beam Finite Element

In this section, the application of the general finite element procedure is illustrated for a finite element model of a Timoshenko beam with rectangular cross section. The model employed here is essentially one developed by J. S. Przemieniecki in 1966.[24] Przemieniecki's model had six degrees of freedom at each node while, for convenience, the model used here constrained all but two of those degrees of freedom, transverse displacement and cross-sectional rotation, to yield planar motion. The geometry of the beam element is given in Fig. 2. The displacement field equation [see Eq. (2)] and the strain-displacement equation [see Eq. (3)] for this finite element are given by Przemieniecki[24] and are, therefore, not shown here. The constitutive law [see Eq. (1)] assumed by Przemieniecki[24] was linear elastic which allowed calculation of mass and stiffness matrices; but, the constitutive law assumed here includes terms with stress related to strain rate to permit calculation of viscous and nonviscous damping matrices. Thus, the constitutive law is written as

$$\begin{Bmatrix} \sigma_{xx} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \gamma_{xz} \end{Bmatrix} + \begin{bmatrix} \bar{E} & 0 \\ 0 & \bar{G} \end{bmatrix} \begin{Bmatrix} \dot{\epsilon}_{xx} \\ \dot{\gamma}_{xz} \end{Bmatrix} + \begin{bmatrix} E^* e(\epsilon_{xx}, \dot{\epsilon}_{xx}) & 0 \\ 0 & G^* g(\gamma_{xz}, \dot{\gamma}_{xz}) \end{bmatrix} \begin{Bmatrix} \dot{\epsilon}_{xx} \\ \dot{\gamma}_{xz} \end{Bmatrix} \tag{11}$$

where E is Young's modulus, G shear modulus, $\bar{E}$ and $\bar{G}$ viscous damping constants, E^* and G^* nonviscous damping constants, and $e(\epsilon_{xx}, \dot{\epsilon}_{xx})$ and $g(\gamma_{xz}, \dot{\gamma}_{xz})$ nonviscous damping functions. Many nonviscous damping functions can be introduced to the model through e and g. For type II damping $e = |\dot{\epsilon}_{xx}|$ and $g = |\dot{\gamma}_{xz}|$. Although, in general, $e = e(\epsilon_{xx}, \dot{\epsilon}_{xx}, \gamma_{xz}, \dot{\gamma}_{xz} \ldots)$, for this model e was chosen as $e(\epsilon_{xx}, \dot{\epsilon}_{xx})$ in keeping with the one-dimensional nature of the Timoshenko beam; this same reasoning also applies to g.

The matrices $[N]$, $[B]$, $[E]$, $[\bar{E}]$, and $[\bar{e}]$ [see Eqs. (1-3)] for this finite element model are then substituted into Eqs. (7-10) to obtain the mass, stiffness, viscous damping, and nonviscous damping matrices for the Timoshenko beam element. Since the mass and stiffness matrices are given by Przemieniecki,[24] only the viscous and nonviscous damping matrices are given here. The element viscous damping matrix is given by

$$[C] = \lambda \begin{bmatrix} 12(1+\alpha\phi) & & & \text{Symmetric} \\ 6L(1+\alpha\phi) & L^2[(4+\phi)(1+\phi)+3\phi(\alpha-1)] & & \\ -12(1+\alpha\phi) & -6L(1+\alpha\phi) & 12(1+\alpha\phi) & \\ 6L(1+\alpha\phi) & L^2[(2-\phi)(1+\phi)+3\phi(\alpha-1)] & -6L(1+\alpha\phi) & L^2[(4+\phi)(1+\phi)+3\phi(\alpha-1)] \end{bmatrix}$$

where $\alpha = \bar{E}G/E\bar{G}$, $\phi = 12EI/Gk_sAL^2$, $\lambda = \bar{E}I/[(1+\phi)^2L^3]$, k_s is the cross-sectional shear coefficient, A the cross-sectional area, I the cross-sectional moment of inertia, and L the element length. For $\bar{E}/\bar{G} = E/G$ the viscous damping matrix is proportional to the stiffness matrix.

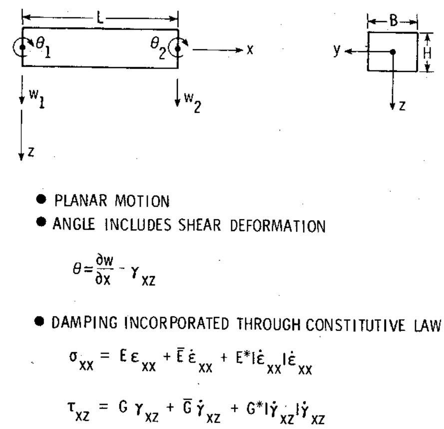

Fig. 2 Timoshenko beam finite element model.

Presented next are the integral expressions for each member $C_{NL,ij}$ of the nonviscous damping matrix in terms of general damping functions e and g. For convenience, nondimensional length and height distances, $\kappa = x/L$ and $\xi = z/H$, respectively, are employed; and y integration over the beam width B has already been done.

$$C_{NL,11} = J_3 \int_{\kappa=0}^{\kappa=1} (6-12\kappa)^2 J_1 \mathrm{d}\kappa + J_2$$

$$C_{NL,12} = LJ_3 \int_{\kappa=0}^{\kappa=1} (6-12\kappa)(4-6\kappa+\phi) J_1 \mathrm{d}\kappa + \frac{L}{2} J_2$$

$$C_{NL,13} = -C_{NL,11}$$

$$C_{NL,14} = LJ_3 \int_{\kappa=0}^{\kappa=1} (6-12\kappa)(2-6\kappa-\phi) J_1 \mathrm{d}\kappa + \frac{L^2}{4} J_2$$

$$C_{NL,22} = L^2 J_3 \int_{\kappa=0}^{\kappa=1} (4-6\kappa+\phi)^2 J_1 \mathrm{d}\kappa + \frac{L^2}{4} J_2$$

$$C_{NL,23} = -C_{NL,12}$$

$$C_{NL,24} = L^2 J_3 \int_{\kappa=0}^{\kappa=1} (4-6\kappa+\phi)(2-6\kappa-\phi) J_1 \mathrm{d}\kappa + \frac{L^2}{4} J_2$$

$$C_{NL,33} = C_{NL,11}$$

$$C_{NL,34} = -C_{NL,14}$$

$$C_{NL,44} = L^2 J_3 \int_{\kappa=0}^{\kappa=1} (2-6\kappa-\phi)^2 J_1 \mathrm{d}\kappa + \frac{L^2}{4} J_2$$

where

$$J_1 = \int_{\xi=-1/2}^{\xi=1/2} e(\epsilon_{xx}, \dot{\epsilon}_{xx}) \xi^2 \mathrm{d}\xi$$

$$J_2 = G^* g(\gamma_{xz}, \dot{\gamma}_{xz}) \frac{k_s A}{L} \left(\frac{\phi}{1+\phi}\right)^2$$

and

$$J_3 = \frac{BE^* H^3}{(1+\phi)^2 L^3}$$

Like the other element matrices this one is also symmetric. When performing the volume integrals for this Timoshenko beam element, the integration of the shear stress for the stiffness and damping matrices is done over a reduced area given by k_s times the cross-sectional area, as required by Timoshenko beam theory.[25] In general, the integrations for $C_{NL,ij}$ are done numerically; but, for type II damping the integrations were done in closed form.[26] Thus, type II damping was employed in the Timoshenko beam problems to be presented in this paper.

Solution Technique

In this section, a solution technique for linear problems, the Newmark method, is modified for application to the set of nonlinear equations Eq. (6b) which result from including nonviscous structural damping in the finite element method. The Newmark method[18] is an implicit time-integration technique which is unconditionally stable for linear problems; and it induces no numerical amplitude decay, though it does exhibit numerical period elongation. The Newmark method, as originally proposed, assumes that the average acceleration over an integration time step Δt is constant. Then, using conditions at the beginning of the time step as initial conditions, displacement and velocity at the end of the time step are predicted using constant-acceleration formulas. Thus, the Newmark equations for displacement and velocity at the end of the time step are

$$\{u(t)\} = \{u(t-\Delta t)\} + \{\dot{u}(t-\Delta t)\}\Delta t + \{\ddot{u}_{\text{avg}}\}(\Delta t^2/2) \tag{12}$$

and

$$\{\dot{u}(t)\} = \{\dot{u}(t-\Delta t)\} + \{\ddot{u}_{\text{avg}}\}\Delta t \tag{13}$$

where

$$\{\ddot{u}_{\text{avg}}\} = [\{\ddot{u}(t)\} + \{\ddot{u}(t-\Delta t\}]/2$$

Equations (12) and (13) are now combined with Eq. (6b) to give three equations with three unknowns: $\{u(t)\}$, $\{\dot{u}(t)\}$, and $\{\ddot{u}(t)\}$. Solving these three equations for an equation with $\{u(t)\}$ as the only unknown results in

$$[\bar{K} + (2/\Delta t)C_{NL}]\{u(t)\} = \{\bar{F}\} + [C_{NL}]\{f(t-\Delta t)\} \tag{14}$$

where

$$[C_{NL}] = [C_{NL}(\{u(t)\},\{\dot{u}(t)\})]$$

$$[\bar{K}] = [K] + (4/\Delta t^2)[M] + (2/\Delta t)[C]$$

$$\{\bar{F}\} = \{F(t)\} + [M]\{(4/\Delta t^2)u(t-\Delta t) + (4/\Delta t)\dot{u}(t-\Delta t)\} + [C]\{f(t-\Delta t)\}$$

and

$$\{f(t-\Delta t)\} = (2/\Delta t)\{u(t-\Delta t)\} + \{\dot{u}(t-\Delta t)\}$$

Equation (14) is of the form

$$[\hat{K}]\{u(t)\} = \{\hat{R}\}$$

where $[\hat{K}]$ is called the effective stiffness matrix and $\{\hat{R}\}$ is called the effective load vector.

Equation (14) is nonlinear and must be solved iteratively. The form of Eq. (14) used in the iteration process is

$$[\bar{K} + (2/\Delta t)C_{NL}(\{u_n(t)\},\{\dot{u}_n(t)\})]\{u_{n+1}(t)\} = \{\bar{F}\} + [C_{NL}(\{u_n(t)\},\{\dot{u}_n(t)\})]\{f(t-\Delta t)\} \tag{15}$$

where n indicates the iteration number. First-approximations $\{u_0(t)\}$ and $\{\dot{u}_0(t)\}$ are calculated using the constant-acceleration formulas

$$\{\dot{u}_0(t)\} = \{\dot{u}(t-\Delta t)\} + \{\ddot{u}(t-\Delta t)\}\Delta t$$

and

$$\{u_0(t)\} = \{u(t-\Delta t)\} + \{\dot{u}(t-\Delta t)\}\Delta t + \{\ddot{u}(t-\Delta t)\}(\Delta t^2/2)$$

These values are used to calculate $[C_{NL}]$ and then $\{u_1(t)\}$ is solved for by Gauss elimination. This new estimate of $\{u(t)\}$ is then used to recalculate $[C_{NL}]$ and $\{u_2(t)\}$ is solved for. Iteration continues until convergence occurs between $\{u_n(t)\}$ and $\{u_{n+1}(t)\}$ such that $|\{u_{n+1}(t)\} - \{u_n(t)\}| / |\{u_n(t)\}| \le \epsilon$ where ϵ is some specified tolerance. Equation (15) is called a tangent stiffness method equation because the nonlinear term in the effective stiffness matrix requires the repeated updating of the effective stiffness matrix during the solution process. This repeated updating also requires repeated triangularizations of the effective stiffness matrix which is expensive if more than a few finite elements are used. To avoid these repeated triangularizations, Eq. (14) is rewritten so that the nonlinear part of the effective stiffness matrix, $2[C_{NL}]/\Delta t$, is factored out and made to appear as a pseudoforce in the effective load vector. Equation (14) thus becomes

$$[\bar{K}]\{u(t)\} = \{\bar{F}\} + [C_{NL}]\{\{f(t-\Delta t)\} - (2/\Delta t)\{u(t)\}\} \tag{16}$$

The form of Eq. (16) used in the iterative solution process is

$$[\bar{K}]\{u_{n+1}(t)\} = \{\bar{F}\} + [C_{NL}(\{u_n(t)\},\{\dot{u}_n(t)\})] \times \{\{f(t-\Delta t)\} - (2/\Delta t)\{u_n(t)\}\} \tag{17}$$

Equation (17) has an effective stiffness matrix which is a constant $[\bar{K}]$ that is triangularized once for all time. Thus, Eq. (17) has the potential for great computational savings over Eq. (15) whose effective stiffness matrix is a function of $\{u_n(t)\}$ and $\{\dot{u}_n(t)\}$ and must, therefore, be repeatedly updated and triangularized. Equation (17) is called a pseudoforce method equation. Past applications of the Newmark method to nonlinear finite element problems have used the tangent stiffness method.[10-13] This is because the past problems considered have been concerned mainly with geometric nonlinearities for which the pseudoforce method has been found to be unstable.[27] Since studies have indicated that the pseudoforce method is effective in problems with material nonlinearities[27,28] and since nonviscous damping is a material nonlinearity, the present study applies the pseudoforce method of Eq. (17). The solution technique used herein is thus referred to as the "pseudoforce Newmark method" to distinguish it from the classical Newmark method used in linear problems. Since this is a unique application of the Newmark method, the accuracy and convergence characteristics were studied and the results are discussed in the following section.

Accuracy and Convergence Characteristics of the Pseudoforce Newmark Method

The accuracy and convergence of the pseudoforce Newmark method for linear and nonlinear initial-condition problems involving a single-degree-of-freedom system and a multi-degree-of-freedom two-element cantilevered beam are discussed in this section. The results presented here are only highlights of an extensive study by Roussos.[26]

Single Degree of Freedom (SDOF)

Accuracy

The accuracy of the pseudoforce Newmark method has been studied for problems with viscous, type I, II, and III damping. As discussed earlier in the paper, Fig. 1 shows the force-displacement-velocity relations for these damping models. Table 1 and Fig. 3 show a comparison of the pseudoforce Newmark method solution with the closed-form solution for a type I damped problem (C_I/critical viscous damping = 0.05). Table 1 shows very good agreement between the two solutions with the difference in times of zero crossing being less than 1% of the undamped period for the first 12 zero crossings. Also, though the results are not presented here, the peak amplitudes were predicted with less than 1% error relative to the closed-form solution for the first 11 peaks. The pseudoforce Newmark solution was also compared to a closed-form solution for viscous damping and to perturbation solutions (method of multiple scales[29]) for type II and III damping, and very good agreement similar to that for type I damping occurred for all comparisons.[26] In all cases, the damping force purposely was made small compared to the inertial and elastic forces in keeping with the nature of structural damping.

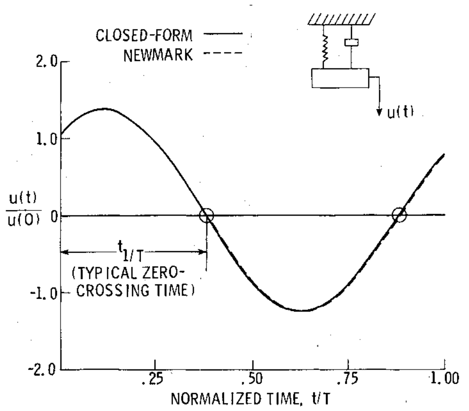

Fig. 3 Comparison of pseudoforce Newmark solution with closed-form solution for type I damping.

Table 1 Zero-crossing comparison of pseudoforce Newmark solution with closed-form solution for type I damped SDOF problem

Zero crossing number, i	Time of zero crossing, t_i/T Pseudoforce Newmark	Closed form	Difference
1	0.380	0.379	0.001
2	0.885	0.882	0.003
3	1.385	1.383	0.002
4	1.884	1.884	0
5	2.384	2.387	0.003
6	2.884	2.889	0.005
7	3.384	3.390	0.006
8	3.883	3.893	0.010
9	4.389	4.394	0.005
10	4.889	4.896	0.007
11	5.391	5.399	0.008
12	5.892	5.900	0.008

Table 2 Convergence characteristics of pseudoforce Newmark method for damping types I, II, and III in SDOF system

Damping force model	Range of initial velocities for swift convergence: Large damping	Small damping
Type I : $C_I \mid u \mid \mathrm{sgn}(\dot{u})$	All velocities	All velocities
Type II : $C_{II} \mid \dot{u} \mid \dot{u}$	Small velocities	All velocities
Type III: $C_{III} u^2 \dot{u}$	No velocities	Small velocities

Convergence Characteristics

The convergence characteristics of the pseudoforce Newmark method have been studied for problems with type I, II, and III damping. The study determined how changes in magnitude of initial velocity and changes in magnitude of damping constant affected the number of iterations needed for convergence. The detailed results of Roussos[26] are qualitatively summarized in Table 2. In general, increases in the degree of the nonlinearity of the damping (going from type I to II to III) increase the number of convergence iterations. Type I damping solutions (convergence tolerance $\epsilon = 0.005$ and 0.001) exhibited convergence in no more than one iteration for both small and large damping constants for all initial velocities tested. (A "small" damping constant is an order of magnitude less than the stiffness, while a "large" damping constant is of the same order of magnitude as the stiffness.) Type II damping solutions ($\epsilon = 0.005$, 0.001, and 0.0001) converged on the average in two iterations for all initial velocities tested with small damping; but had similar convergence only for small initial velocities with large damping. (Small initial velocities are initial velocities that result in small damping forces throughout most of the motion.) Type III damping solutions ($\epsilon = 0.005$ and 0.001) exhibited convergence in two or less iterations for small damping constant and small initial velocities, while needing many more iterations and sometimes diverging for other damping sizes and initial velocities.

Cantilevered Beam

This section is an investigation of the accuracy and efficiency of the classical and pseudoforce Newmark methods in solving a two-element (four degrees of freedom) cantilevered Timoshenko beam problem with no damping, viscous damping, and type II damping. The initial condition used for all the problems was a tip displacement of 2.54 cm (1.0 in.). The corresponding initial midpoint displacement and tip and midpoint cross section rotations were calculated using static beam theory. All initial velocities were set equal to zero. The physical parameters of the cantilevered beam were beam length $l = 152.4$ cm (60.0 in.) ($l = 2L$); beam height $H = 5.08$ cm (2.0 in.); beam width $B = 5.08$ cm (2.0 in.); radius of gyration = 1.47 cm (0.58 in.); Young's modulus $E = 21 \times 10^{10}$ Pa (30×10^6 lb/in.^{2}); shear modulus $G = 8 \times 10^{10}$ Pa (12×10^6 lb/in.^{2}); mass density $\rho = 0.0078$ kg/cm^3 (0.00073 lb-s^2/in.^{4}); and first-mode period $T = 0.055$ s.

As a verification and support of the results obtained using the classical and pseudoforce Newmark methods, Gear method numerical solutions were also obtained using the International Mathematical and Statistical Library (IMSL)[30] subroutine DVOGER. The Gear method[31] is a numerical-integration technique for solving simultaneous ordinary differential equations, but it is inefficient for finite element problems with more than a few finite elements. The integration time step used by DVOGER was continuously updated by the subroutine as it solved a problem.

A comparison of the pseudoforce Newmark solution ($\Delta t = 5 \times 10^{-5}$ s) with the classical Newmark solution ($\Delta t = 5 \times 10^{-5}$ s) for the case of viscous damping (damping constants $\bar{E}/E = \bar{G}/G = 0.000040$ s) was obtained as a verification that the accuracy of the classical Newmark method is retained in the pseudoforce representation for multi-degree-of-freedom problems. Agreement between the Newmark solutions was excellent with times of zero crossing differing by less than 0.15% of T after four zero crossings.

To determine the accuracy of the classical Newmark method relative to the Gear method, a problem with no damping was solved. Newmark method solutions for time steps of $\Delta t = 25 \times 10^{-5}$ s and 5×10^{-5} s were compared with a Gear method solution using an average Δt of about 5×10^{-5}

Table 3 Zero-crossing comparison of pseudoforce Newmark and Gear method solutions for endpoint displacement of cantilevered beam with type II damping

Zero crossing number, i	Time of zero crossing, t_i/T Pseudoforce Newmark	Gear method	Difference
1	0.247	0.249	0.002
2	0.743	0.744	0.001
3	1.238	1.249	0.011
4	1.732	1.749	0.017

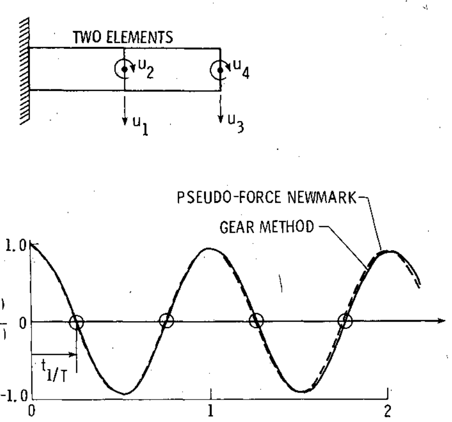

Fig. 4 Comparison of pseudoforce Newmark and Gear method type II damped solutions.

s. Assuming the Newmark solution with the smaller Δt was the more accurate of the two Newmark solutions; the results indicated that the less accurate Newmark solution obtained the same accuracy as the Gear method solution in about 80% less central processing unit (CPU) time while using an integration time step about five times bigger; and the more accurate Newmark solution, which used about the same integration time step as the Gear method solution, was more accurate than the Gear method solution while using about 35% less CPU time.

Clearly, the classical Newmark method is a more efficient time-integration technique than the Gear method for the problem just discussed. The results did demonstrate that the Gear method solution is indeed an effective, though inefficient, check for the pseudoforce Newmark solutions.

Finally, to determine the accuracy of the pseudoforce Newmark method in a multi-degree-of-freedom system for a problem with nonviscous damping, comparison of pseudoforce Newmark ($\Delta t = 10 \times 10^{-5}$ s) and Gear method (average $\Delta t = 44 \times 10^{-5}$ s) solutions for a problem with type II damping ($E^*/E = G^*/G = 0.002$ s^2) was obtained. A comparison of times of zero crossing is shown in Table 3 and time history plots are shown in Fig. 4. The results showed that good agreement occurred between the Gear method and pseudoforce Newmark solutions with the difference in times of zero crossing being less than 1.7% of T after four zero crossings. The CPU times for the pseudoforce Newmark and Gear method solutions were about 16 and 230 s, respectively. Judging by the comparative sizes of the integration time steps and by the previous results presented (which showed the classical Newmark method to be more accurate than the Gear method and showed the pseudoforce Newmark method retained the accuracy of the classical Newmark method), the pseudoforce Newmark solution for this type II damped problem was probably more accurate than the Gear method solution while being obtained in 90% less CPU time.

The results presented here certainly demonstrate that the pseudoforce Newmark method is an efficient, accurate, and, thereby, feasible solution technique for two-element cantilevered beam problems with small damping (small E^*, G^* relative to E, G).

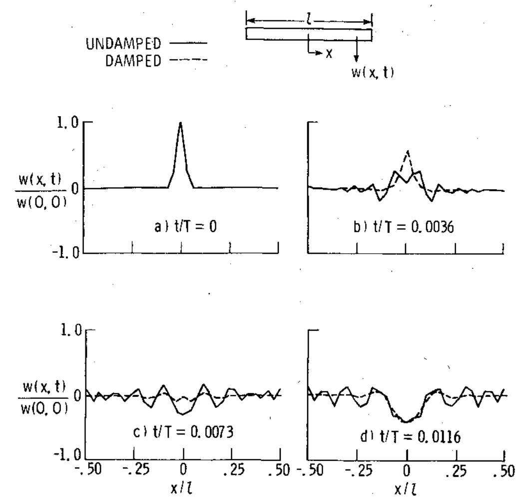

Fig. 5 Undamped and type II damped free-free Timoshenko beam; T = first mode period.

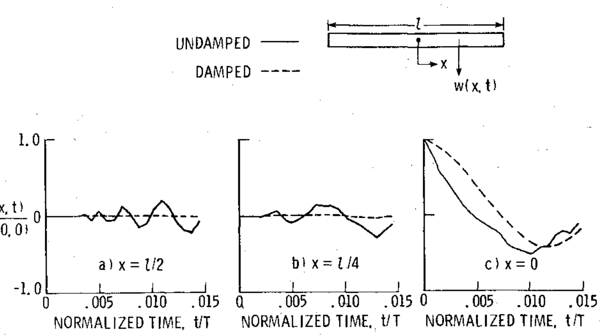

Fig. 6 Undamped and type II damped time histories of free-free Timoshenko beam.

Free-Free Timoshenko Beam Problem

This paper has developed a finite element model for incorporating nonviscous damping in a structure and a solution technique for the resulting nonlinear equations of motion. The ultimate interest lies in applying these two developments to determine the effect of damping on the motion of a structure. This section of the paper applies the finite element model and the solution technique to show the effect of type II damping on the motion of a free-free Timoshenko beam subjected to an initial displacement condition. The solution of this simple problem is meant to be a first approximation of the effect of structural damping on the motion of a long, flexible member of a large outer-space structure.

The initial displacement condition (initial velocities are set equal to zero and there were no external forces) consists of a cosine-shaped disturbance of unit height centered about the beam midspan. With r_D denoting the ratio of the disturbance length l_D to the beam length l and x being measured from the middle of the beam, the initial displacement equation is

$$\text{initial displacement} = \tfrac{1}{2}\left[1 + \cos\left(\frac{2\pi x}{r_D l}\right)\right]; \quad \frac{-l_D}{2} < x < \frac{l_D}{2}$$

$$= 0; \quad \text{all other } x$$

(see Fig. 5a). The initial cross section rotations were defined so that they were perpendicular to the neutral axis, thus, the equation for initial rotation is

$$\text{initial rotation} = \frac{-\pi}{r_D l}\sin\left(\frac{2\pi x}{r_D l}\right); \quad \frac{-l_D}{2} < x < \frac{l_D}{2}$$

$$= 0; \qquad \text{all other } x$$

The problem solved has $r_D = 0.1$. The beam parameters and material properties were the same as those for the cantilevered beam problem solved in the previous section except for beam length $l = 1219.2$ cm (480.0 in.) which resulted in a longer first-mode period T of 0.55 s. The number of elements used was 20, and the time increment Δt was 4×10^{-6} s.

The effects of type II damping on the motion of the beam are displayed graphically in Figs. 5 and 6. Figure 5 shows the transient response of the beam at four selected times without damping and with type II damping. Figure 6 shows undamped and damped time histories for the endpoint ($x = l/2$), quarter point ($x = l/4$), and midpoint ($x = 0$) of the beam.

As seen in Figs. 5 and 6, structural damping, as expected, greatly affected the high-frequency components of beam motion. In the undamped solution, the initial displacement disperses as it moves outward from the center of the beam. The undamped initial displacement disperses so much of its energy outward that by the time the initial displacement inverted at about $t/T = 0.01$, its midpoint displacement $w(0,t)$ was only about $-0.51\ w(0,0)$. A perfectly inverted initial displacement would have had a midpoint displacement of $-w(0,0)$. This dispersion is characteristic of hyperbolic systems such as the Timoshenko beam. However, in the damped solution, the dispersive motion that was displayed in the undamped solution is almost nonexistent. At the final time investigated, Fig. 6 shows that the endpoint has not yet noticeably moved while the quarter point has just begun to move. This is in contrast with the oscillations experienced by the ends of the beam for the undamped case. In effect, with damping the ends of the beam do not respond to the initial displacement at the center of the beam; while with no damping, the ends do eventually respond to the initial displacement in the center of the beam. Even though dispersion such as that shown in the undamped case can be caused by spatial discretization error, Belytschko and Mindle[32] have shown that little or no error occurs in the problem at hand because of the disturbance wavelength being almost 200 times the radius of gyration. Clearly, the proper modeling of structural damping is important in the design of control systems of large space structures insofar as the location and spacing of sensors and controllers is concerned.

Concluding Remarks

This paper has addressed the concern of predicting the dynamic response of structures having damping dominated by internal structural damping, both viscous and nonviscous. Structural damping has been modeled as a relationship between stress, strain, and strain rate in a material constitutive law. This constitutive law has been used in conjunction with the finite element method to develop general integral expressions for viscous and nonviscous damping matrices. To solve the set of nonlinear equations resulting when nonviscous damping is present, a solution technique was developed by modifying the Newmark method to accommodate an iterative solution and to treat the nonviscous damping as a pseudoforce. The solution technique was checked for accuracy and convergence in single- and multiple-degree-of-freedom problems and was found to be accurate and efficient for initial-condition problems with small nonviscous damping. Finally, the analysis approach was demonstrated by determining the effects of a specific nonviscous damping model on the transient motion of a free-free Timoshenko beam.

Thus, the modeling of structural damping through strain and strain rate terms in the constitutive law of a material has been shown to be an effective analytical tool because it readily leads to the calculation of damping matrices in the finite element method which yields equations of motion that can be solved to investigate the effect of damping. However, the practicality of this approach is yet to be verified experimentally.

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